Lower Bounds for a Class of Kostka Numbers

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ABSTRACT

A simple proof of a lower bound on the number of $2m \times 2m$ matrices with 0,1 entries and each of whose rows and columns adds to the fixed sum m is presented. In fact, it is shown that for any fixed $0 < \lambda < 1/2$ the number of such matrices is asymptotically at least $\binom{2m}{m}^{m+\lambda m}$. The inductive proof employed in the present paper might also turn out to be useful in obtaining lower bounds for other types of Kostka numbers.

1. Introduction.

0,1 matrices, i.e. matrices whose only entries are the bits 0 and 1, occur as the characteristic functions of finite binary relations. Thus, let $R \subseteq N \times E$ be a binary relation on the set $N \times E$, where $N = \{v_1,...,v_m\}$ and $E = \{e_1,...,e_n\}$. Define

$$a_{i,j} = \begin{cases} 1 & \text{if } R(v_i,e_j) \\ 0 & \textit{otherwise.} \end{cases}$$

Then the $m \times n$ matrix $A_R = (a_{i,j})$ contains a complete description of the relation R. Here are some examples from graph theory. If N is the set of nodes, E is the set of edges of a graph G and $R(v_i,e_j)$ if and only if v_i is incident to e_j, then A_R is called the *incident matrix* of $G = (N,E)$. If $N = E$ is the set of nodes of a graph G and $R(v_i,v_j)$ if and only if v_i is adjacent to v_j, then A_R is called the *adjacency matrix* of $G = (N,E)$. Some information regarding the structure of such *incidence matrices* can be found in [10].

For each m,k let $\left\{ {m \atop k} \right\}$ be the number of different $m \times m$ square matrices with 0,1 entries such that the sum of the entries of each row and each column is exactly k.

A lot of interesting results are known about $\{{m \atop k}\}$, when k is small with respect to m. For fixed k, [2] gives the general asymptotic result

$$\{{m \atop k}\} \approx \frac{(km)!}{(k!)^{2m}} \cdot exp\left(\frac{-(k-1)^2}{2}\right),$$

as m tends to ∞. If $k = 2$, [3] provides the recursive formula

$$\{{m \atop 2}\} = \binom{m}{2} \cdot [2\{{m-1 \atop 2}\} + (m-1)\{{m-2 \atop 2}\}].$$

A general asymptotic formula is also given in [7], in which it is stated that for any $0 < c < 1$ and any $1 < k < c \cdot \sqrt{\log m}$,

$$\{{m \atop k}\} = \frac{(mk)!}{(k!)^{2m}} \cdot \left[\exp\left(-\frac{(k-1)^2}{2}\right) + 0(m^{-1+c^2/2})\right],$$

as m tends to ∞.

Further asymptotic results are also known for small k, e.g. see [9] for $k = 3$, and [8] for $k = 0((\log n)^{1/4} - \epsilon)$.

For larger k the result of [5] is known, which can be stated as follows: if $k < cm$, for $c < 1/6$, then uniformly

$$\{{m \atop k}\} = \frac{(mk)!}{(k!)^{2m}} \cdot \exp\left(-\frac{(k-1)^2}{2} + 0\left(\frac{k^3}{m}\right)\right).$$

For more information see [4].

In addition, there has been a lot of work regarding an exact formula for $\{{m \atop k}\}$. Such a rather unmanageable formula is given on page 235 of [2]. Additional work, which is influenced by the theory of group characters, can be found in [11]. A necessary and sufficient condition for the existence of 0,1 matrices with prescribed row and column sums is provided by the Gale-Ryser theorem (see [10]). The numbers $\{{m \atop k}\}$ are also known in the literature as Kostka number (see [3], [6]).

However, very little seems to be known on $\{{m \atop k}\}$ when k is close to $m/2$, e.g. $\{{2m \atop m}\}$ or even $\{{2m \atop m-1}\}$. The purpose of the present paper is to give a simple proof of a lower bound on the size of $\{{2m \atop m}\}$. This is done by formulating an inductive property that must be satisfied by all potential 0,1 matrices. Moreover, if this property is satisfied by the first k rows of the matrix ($k < m$) then it is possible to extend the matrix by adding one more row in such a way that the property is still satisfied by the first $k+1$ rows of the matrix. This makes it possible to recursively construct such matrices and hence give a lower bound on $\left\{ {2m \atop m} \right\}$.

There is a lot of interest in determining the value of $\left\{ {2m \atop m} \right\}$. In fact, it appears that there is work in progress by Odlyzko, Zagier and McKay in order to determine the exact asymptotic behaviour of $\left\{ {2m \atop m} \right\}$ (personal communication with McKay).

2. The Counting Arguments.

An obvious way to obtain a lower bound for $\left\{ {2m \atop m} \right\}$ is to count all possible arrangements of the bits 0,1 on the top half of the matrix and then reflect the resulting configuration to the bottom half. In fact, the following result can be shown.

Theorem 2.1. *For all $m > 0$,*

$$\binom{2m}{m}^m \leq \left\{ {2m \atop m} \right\} \leq \binom{2m}{m}^{2m-1}.$$

Proof. Let a $2m \times 2m$ square matrix be given. Split the matrix into two parts. The top m rows constitute the *top* part and the bottom m rows constitute the *bottom* part. For any given row of the top part there are $\binom{2m}{m}$ different ways to arrange the bits 1 and 0 inside this row and in such a way that the sum of its entries is m. In particular, there are $\binom{2m}{m}^m$ different ways to arrange the bits 1 and 0 inside the top part of a matrix in such a way that the sum of the entries of each row is m. It remains to fill the bottom part of the matrix. This is done by adjoining to the bottom half the (0,1) complement of the top half of the matrix. Now it will be shown that the sum of the entries of each column must be equal to m. Indeed, let C be an arbitrary column with entries $c_1,\ldots,c_{2m}$, where c_i is the entry of the ith row of C. By construction of the matrix for each $i = 1,\ldots,m$, $c_i + c_{i+m} = 1$. Hence, $\sum_{i=1}^{2m} c_i = m$, as desired. This establishes the desired lower bound.

The upper bound is much easier to prove. There are $2m$ possible rows in any matrix. However, the sum property satisfied by the matrix columns implies that the entries of any row can be computed from the entries of the remaining $2m-1$ rows. □

Here is a table comparing the values of $\left\{ {2m \atop m} \right\}$ with the approximate values of the upper and lower bounds of theorem 2.1, for $m = 1,2,\ldots,10$.

In addition, the third column of the table gives the approximate lower bound of theorem 2.4, i.e. $\binom{2m}{m}^{3m/2}$. L. Meertens has written a computer program to determine the values of $\left\{ {2m \atop m} \right\}$, from which the values below corresponding to $m \leq 4$ are taken. For $m > 4$ the values are taken from the tables in [4].

m	$\binom{2m}{m}^m$	$\binom{2m}{m}^{3m/2}$	$\left\{ {2m \atop m} \right\}$	$\binom{2m}{m}^{2m-1}$
1	2	2.82	2	2
2	36	216	90	216
3	$8.00 \cdot 10^{3}$	$7.15 \cdot 10^{5}$	$2.97 \cdot 10^{5}$	$3.20 \cdot 10^{6}$
4	$2.40 \cdot 10^{7}$	$1.18 \cdot 10^{11}$	$1.17 \cdot 10^{11}$	$8.24 \cdot 10^{12}$
5	$1.02 \cdot 10^{12}$	$1.02 \cdot 10^{18}$	$6.73 \cdot 10^{18}$	$4.19 \cdot 10^{21}$
6	$6.22 \cdot 10^{17}$	$4.91 \cdot 10^{26}$	$6.41 \cdot 10^{28}$	$4.19 \cdot 10^{32}$
7	$5.61 \cdot 10^{24}$	$1.33 \cdot 10^{37}$	$1.09 \cdot 10^{41}$	$9.16 \cdot 10^{45}$
8	$7.53 \cdot 10^{32}$	$2.07 \cdot 10^{49}$	$3.48 \cdot 10^{55}$	$4.40 \cdot 10^{61}$
9	$1.52 \cdot 10^{42}$	$1.87 \cdot 10^{63}$	$2.19 \cdot 10^{72}$	$4.74 \cdot 10^{79}$
10	$4.63 \cdot 10^{52}$	$9.98 \cdot 10^{78}$	$2.79 \cdot 10^{91}$	$1.16 \cdot 10^{100}$

A less trivial approach to obtain a better lower bound is via the method of *filling correct patterns*. The idea is as follows. Begin filling up the rows of a $2m \times 2m$ matrix starting from the top and moving downwards. After k rows have been filled call the resulting pattern *correct* if:

(i) each of the k rows has exactly m 1's, and

(ii) each of the $2m$ columns has $\leq m$ 1's and $\leq m$ 0's.

Then it can be shown that

Lemma 2.2. *Every correct pattern of $k < 2m$ rows can be extended to a correct pattern of $k+1$ rows.*

Proof. Let a correct pattern of $k < 2m$ be given. Define

$$N_0(i) = |\{(j,i) \colon j \leq k,\, a_{j,i} = 0\}|,$$

$$N_1(i) = |\{(j,i) \colon j \leq k,\, a_{j,i} = 1\}|,$$

i.e. the number of 0's and 1's respectively in the ith column. So, by definition of *correct patterns* we have that for all $i = 1,\ldots,2m$,

$$N_0(i) \leq m, \quad N_1(i) \leq m. \tag{1}$$

Further, define $M_0 = |\{i \colon N_0(i) = m\}|$ and $M_1 = |\{i \colon N_1(i) = m\}|$ as the number of columns having already their maximal share of 0's and 1's respectively, viz. m. Note that

$$N_0(i) + N_1(i) = k, \tag{2}$$

for $i = 1,\ldots,2m$, and

$$\sum_{i=1}^{2m} N_0(i) = \sum_{i=1}^{2m} N_1(i) = mk. \tag{3}$$

Claim: $M_0, M_1 \leq m$.

Proof of the claim: Assume on the contrary that $M_1 = m + r$, for some $r \geq 1$. The proof of $M_0 \leq m$ is entirely similar. (Note that this implies $k \geq m$.) Then

$$\sum_{i=1}^{2m} N_1(i) \geq (m+r)m + (m-r)(k-m),$$

since there are $m + r$ columns with m occurrences of 1 and the remaining $m - r$ columns have by (1), (2) at least $k - m$ occurrences of 1. Therefore it follows from (3) that

$$mk \geq (m+r)m + (m-r)(k-m)$$

or equivalently $k \geq 2m$, contradicting the assumption $k < 2m$. This proves the claim.

From the claim the result follows immediately. Indeed, put

$$a_{k+1,i} = \begin{cases} 0 \text{ if } N_1(i) = m \\ 1 \text{ if } N_0(i) = m \end{cases}$$

and choose the remaining entries in the $(k+1)$-th row such that there are m occurrences of 1 and m occurrences of 0. Obviously such a pattern of $k + 1$ rows is still correct, which completes the proof of the lemma. □

Based on this last lemma the counting argument runs as follows. Let $0 < c < m$ be a fixed constant integer. Consider $2m \times 2m$ matrices with only the first m rows filled in such a way that each row has exactly m 0's and exactly m 1's. Let $A(c)$ be the number of such matrices each of whose columns has at most $m - c$ 0's and $m - c$ 1's. (Note that $A(c)$ is positive only if $2c \leq m$.) This requirement means that rows $m+1,\ldots,m+c$ can be filled in an arbitrary manner (of course with the usual restriction that each row has m 0's and m 1's), while still retaining a correct pattern. So, to each of these $A(c)$ half-filled matrices there correspond $\binom{2m}{m}^c$

matrices, **which** can be filled to a full $2m \times 2m$ matrix forming a correct **pattern.** Hence, it is clear that

$$\left\{{2m \atop m}\right\} \geq A(c)\cdot\binom{2m}{m}^c.$$

It remains to determine a lower bound on $A(c)$. Let $A_i(c)$ be the number of matrices with rows $1,\ldots,m$ filled in such a way that at least one of the columns has less than c i's, where $i = 0,1$. It is clear that

$$\binom{2m}{m}^m - A(c) \leq A_0(c) + A_1(c) = 2A_0(c).$$

Hence it is enough to determine an upper bound on the quantity $A_0(c)$. To this end, let $B_0(c)$ be the number of half-way correctly filled matrices with *first* column containing less than c 0's. Clearly, $A_0(c) \leq 2m\cdot B_0(c)$. Further, let $B_{0,j}$ be the number of half-way correctly filled matrices with the *first* column containing exactly j 0's, where $j = 0,\ldots,c-1$. So

$$B_0(c) = B_{0,0} + \cdots + B_{0,c-1}.$$

Now it is easy to see that

$$B_{0,j} = \binom{m}{j}\cdot\binom{2m-1}{m-1}^{m-j}\cdot\binom{2m-1}{m}^j = \binom{m}{j}\cdot\binom{2m-1}{m-1}^m.$$

Hence, noting that

$$\binom{2m-1}{m-1}^m = \binom{2m}{m}^m\cdot 2^{-m}$$

and putting

$$F_m(c) = \binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{c-1}$$

we obtain

$$A_0(c) \leq 2m\cdot F_m(c)\cdot\binom{2m}{m}^m\cdot 2^{-m}.$$

It follows that

$$\left\{{2m \atop m}\right\} \geq \left[\binom{2m}{m}^m - \frac{4m\cdot F_m(c)\cdot\binom{2m}{m}^m}{2^m}\right]\cdot\binom{2m}{m}^c.$$

Hence, the following theorem has been proved.

Theorem 2.3. *For all $m > 0$ and any integer $0 < c < m/2$ the following inequality holds:*

$$\left\{{2m \atop m}\right\} \geq \left[1 - \frac{4m\cdot F_m(c)}{2^m}\right]\cdot\binom{2m}{m}^{m+c}. \quad \Box$$

3. The Lower Bounds.

Finally, it remains to determine values of c for which theorem 2.3 provides nontrivial lower bounds to $\left\{{2m \atop m}\right\}$. The following useful formula on sums of binomial coefficients can be found in page 76 of [2]: for $k < m$,

$$\sum_{i=0}^{k}\binom{m}{i} = (m-k)\cdot\binom{m}{k}\cdot\int_1^2 t^k(2-t)^{m-k-1}dt.$$

(It can be proved easily using integration by parts.) However, for $t \geq 1$, $t(2-t) \leq 1$. Hence, it follows that for $k < m/2$,

$$\sum_{i=0}^{k}\binom{m}{i} \leq (m-k)\cdot\binom{m}{k}\cdot\int_1^2 (2-t)^{m-2k-1}dt = \frac{m-k}{m-2k}\cdot\binom{m}{k}.$$

In addition, the following formula gives a generalization of the binomial coefficient $\binom{m}{x}$ when $x \leq m$ is an arbitrary nonnegative real:

$$\binom{m}{x} = \frac{m!}{\Gamma(x+1)\cdot\Gamma(m-x+1)},$$

where Γ is the gamma function. However, it is easy to show that the function $\binom{m}{x}$ is nondecreasing for $0 < x < m/2$, e.g. this can be proved easily using the identity

$$\Gamma(x) = \lim_{n\to\infty}\frac{n^x\cdot n!}{x(x+1)\ldots(x+n)},$$

which is stated as formula (10) in page 13 of [1]. It follows from the monotonicity of $\binom{m}{x}$, that for any $0 < \lambda < 1/2$,

$$\sum_{i=0}^{\lfloor\lambda m\rfloor}\binom{m}{i} \leq \frac{m-\lfloor\lambda m\rfloor}{m-2\lfloor\lambda m\rfloor}\cdot\binom{m}{\lfloor\lambda m\rfloor} \leq \frac{m-\lfloor\lambda m\rfloor}{m-2\lfloor\lambda m\rfloor}\cdot\binom{m}{\lambda m}.$$

Using Stirling's formula

$$\Gamma(x+1) \approx x^x\cdot e^{-x}\cdot\sqrt{2\pi x}$$

for approximating the gamma function (see [1]) it is easy to see that for positive $\lambda < 1$,

$$\binom{m}{\lambda m} \approx \frac{2^{mH(\lambda)}}{\sqrt{2\pi\lambda(1-\lambda)m}},$$

where $H(\lambda) = -\lambda\log_2\lambda - (1-\lambda)\log_2(1-\lambda)$ is the so called *binary entropy* of λ. Hence, for $\lambda < 1/2$ and m large enough it is asymptotically true that

$$\sum_{i=0}^{\lfloor\lambda m\rfloor}\binom{m}{i} \leq \frac{1-\lambda+(\lambda m-\lfloor\lambda m\rfloor)/m}{1-2\lambda+2(\lambda m-\lfloor\lambda m\rfloor)/m}\cdot\binom{m}{\lambda m}$$

$$\leq \frac{1-\lambda+(\lambda m-\lfloor\lambda m\rfloor)/m}{1-2\lambda+2(\lambda m-\lfloor\lambda m\rfloor)/m}\cdot\frac{1}{\sqrt{2\pi\lambda(1-\lambda)}}\cdot\frac{1}{\sqrt{m}}\cdot 2^{mH(\lambda)}.$$

Consequently, asymptotically, the lower bound of theorem 2.3 becomes

$$\left\{{2m \atop m}\right\} \geq \left[1-\frac{\sqrt{m}\cdot C(m,\lambda)}{2^{m(1-H(\lambda))}}\right]\cdot\binom{2m}{m}^{m+\lfloor\lambda m\rfloor+1},$$

where for each fixed λ

$$C(m,\lambda) = 4\cdot\frac{1-\lambda+\dfrac{\lambda m-\lfloor\lambda m\rfloor}{m}}{1-2\lambda+2\cdot\dfrac{\lambda m-\lfloor\lambda m\rfloor}{m}}\cdot\frac{1}{\sqrt{2\pi\lambda(1-\lambda)}},$$

tends to the quantity

$$C(\lambda) = 4\cdot\frac{1-\lambda}{1-2\lambda}\cdot\frac{1}{\sqrt{2\pi\lambda(1-\lambda)}},$$

as m tends to ∞. Moreover, for fixed $0 < \lambda < 1/2$, the factor

$$E(m,\lambda) = 1 - \frac{\sqrt{m}\cdot C(\lambda)}{2^{m(1-H(\lambda))}}$$

converges to 1 as m tends to ∞. Hence, asymptotically, the following inequality has been proved:

$$\left\{{2m \atop m}\right\} \geq E(m,\lambda)\cdot\binom{2m}{m}^{m+\lfloor\lambda m\rfloor+1}.$$

In fact, without loss of generality, this last inequality can be stated as follows without mentioning $\lfloor\lambda m\rfloor$ at all.

Theorem 3.1. *For all $0 < \lambda < 1/2$, it is asymptotically true that*

$$\left\{ {2m \atop m} \right\} \geq \binom{2m}{m}^{m+\lambda m}. \quad \square$$

Remark. By properly modifying the above proof the interested reader should have no difficulty in formulating the result of theorem 3.1 as a proper inequality, which does not refer to *asymptotics* at all.

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